FOUR GAMES ON BOOLEAN ALGEBRAS

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Abstract

The games $\mathcal{G}_2$ and $\mathcal{G}_3$ are played on a complete Boolean algebra $\mathbb{B}$ in ω-many moves. At the beginning White picks a non-zero element p of $\mathbb{B}$ and, in the n-th move, White picks a positive $p_n < p$ and Black chooses an $i_n \in \{0,1\}$. White wins $\mathcal{G}_2$ iff $\liminf p_n^{i_n} = 0$ and wins $\mathcal{G}_3$ iff $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} p_n^{i_n} = 0$. It is shown that White has a winning strategy in the game $\mathcal{G}_2$ iff White has a winning strategy in the cut-and-choose game $\mathcal{G}_{\mathrm{c\&c}}$ introduced by Jech. Also, White has a winning strategy in the game $\mathcal{G}_3$ iff forcing by $\mathbb{B}$ produces a subset R of the tree ${}^{<\omega}2$ containing either $\varphi^\frown 0$ or $\varphi^\frown 1$, for each $\varphi \in {}^{<\omega}2$, and having unsupported intersection with each branch of the tree ${}^{<\omega}2$ belonging to V. On the other hand, if forcing by $\mathbb{B}$ produces independent (splitting) reals then White has a winning strategy in the game $\mathcal{G}_3$ played on $\mathbb{B}$. It is shown that $\diamondsuit$ implies the existence of an algebra on which these games are undetermined.

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1. Introduction

In [3] Jech introduced the cut-and-choose game $\mathcal{G}_{\mathrm{c\&c}}$, played by two players, White and Black, in ω-many moves on a complete Boolean algebra $\mathbb{B}$ in the following way. At the beginning, White picks a non-zero element $p \in \mathbb{B}$ and, in the n-th move, White picks a non-zero element $p_n < p$ and Black chooses an $i_n \in \{0,1\}$. In this way two players build a sequence $\langle p, p_0, i_0, p_1, i_1, \ldots\rangle$ and White wins iff $\bigwedge_{n\in\omega} p_n^{i_n} = 0$ (see Definition 1).

A winning strategy for a player, for example White, is a function which, on the basis of the previous moves of both players, provides "good" moves for White such that White always wins. So, for a complete Boolean algebra $\mathbb{B}$ there are three possibilities: 1) White has a winning strategy; 2) Black has a winning strategy or 3) none of the players has a winning strategy. In the third case the game is said to be undetermined on $\mathbb{B}$.

The game-theoretic properties of Boolean algebras have interesting algebraic and forcing translations. For example, according to [3] and well-known facts concerning infinite distributive laws we have the following results.

Theorem 1. (Jech) For a complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) White has a winning strategy in the game $\mathcal{G}_{\mathrm{c\&c}}$;

(b) The algebra $\mathbb{B}$ does not satisfy the $(\omega, 2)$-distributive law;

(c) Forcing by $\mathbb{B}$ produces new reals in some generic extension;

(d) There is a countable family of 2-partitions of the unity having no common refinement.

Also, Jech investigated the existence of a winning strategy for Black and using $\diamondsuit$ constructed a Suslin algebra in which the game $\mathcal{G}_{\mathrm{c\&c}}$ is undetermined. Moreover in [6] Zapletal gave a ZFC example of a complete Boolean algebra in which the game $\mathcal{G}_{\mathrm{c\&c}}$ is undetermined.

Several generalizations of the game $\mathcal{G}_{\mathrm{c\&c}}$ were considered. Firstly, instead of cutting of p into two pieces, White can cut into λ pieces and Black can choose more than one piece (see [3]). Secondly, the game can be of uncountable length so Dobrinen in [1] and [2] investigated the game $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ played in κ-many steps in which White cuts into λ pieces and Black chooses less then μ of them.

In this paper we consider three games $\mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$ obtained from the game $\mathcal{G}_{\mathrm{c\&c}}$ (here denoted by $\mathcal{G}_1$) by changing the winning criterion in the following way.

Definition 1. The games $\mathcal{G}_k$, $k \in \{1, 2, 3, 4\}$, are played by two players, White and Black, on a complete Boolean algebra $\mathbb{B}$ in ω-many moves. At the beginning White chooses a non-zero element $p \in \mathbb{B}$. In the n-th move White chooses a $p_n \in (0, p)_{\mathbb{B}}$ and Black responds choosing p_n or $p \setminus p_n$ or, equivalently, picking an $i_n \in \{0, 1\}$ chooses $p_n^{i_n}$, where, by definition, $p_n^0 = p_n$ and $p_n^1 = p \setminus p_n$. White wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ in the game

$\mathcal{G}_1$ if and only if $\bigwedge_{n\in\omega} p_n^{i_n} = 0$;

$\mathcal{G}_2$ if and only if $\bigvee_{k\in\omega} \bigwedge_{n\geq k} p_n^{i_n} = 0$, that is $\liminf p_n^{i_n} = 0$;

$\mathcal{G}_3$ if and only if $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} p_n^{i_n} = 0$;

$\mathcal{G}_4$ if and only if $\bigwedge_{k\in\omega} \bigvee_{n\geq k} p_n^{i_n} = 0$, that is $\limsup p_n^{i_n} = 0$.

In the following theorem we list some results concerning the game $\mathcal{G}_4$ which are contained in [5].

Theorem 2. (a) White has a winning strategy in the game $\mathcal{G}_4$ played on a complete Boolean algebra $\mathbb{B}$ iff forcing by $\mathbb{B}$ collapses $\mathfrak{c}$ to ω in some generic extension.

(b) If $\mathbb{B}$ is the Cohen algebra $\mathrm{r.o.}({}^{<\omega}2, \supseteq)$ or a Maharam algebra (i.e. carries a positive Maharam submeasure) then Black has a winning strategy in the game $\mathcal{G}_4$ played on $\mathbb{B}$.

(c) $\diamondsuit$ implies the existence of a Suslin algebra on which the game $\mathcal{G}_4$ is undetermined.

The aim of the paper is to investigate the game-theoretic properties of complete Boolean algebras related to the games $\mathcal{G}_2$ and $\mathcal{G}_3$. So, Section 2 contains some technical results, in Section 3 we consider the game $\mathcal{G}_2$, Section 4 is devoted to the game $\mathcal{G}_3$ and Section 5 to the algebras on which these games are undetermined.

Our notation is standard and follows [4]. A subset of ω belonging to a generic extension will be called supported iff it contains an infinite subset of ω belonging to the ground model. In particular, finite subsets of ω are unsupported.

2. Winning a play, winning all plays

Using the elementary properties of Boolean values and forcing it is easy to prove the following two statements.

Lemma 1. Let $\mathbb{B}$ be a complete Boolean algebra, $\langle b_n : n \in \omega \rangle$ a sequence in $\mathbb{B}$ and $\sigma = \{\langle \check{n}, b_n \rangle : n \in \omega\}$ the corresponding name for a subset of ω. Then
(a) $\bigwedge_{n\in\omega} b_n = \|\sigma = \check{\omega}\|$;
(b) $\liminf b_n = \|\sigma \text{ is cofinite}\|$;
(c) $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} b_n = \|\sigma \text{ is supported}\|$;
(d) $\limsup b_n = \|\sigma \text{ is infinite}\|$.

Lemma 2. Let $\mathbb{B}$ be a complete Boolean algebra, $p \in \mathbb{B}^+$, $\langle p_n : n \in \omega \rangle$ a sequence in $(0, p)_\mathbb{B}$ and $\langle i_n : n \in \omega \rangle \in {}^\omega 2$. For $k \in \{0, 1\}$ let $S_k = \{n \in \omega : i_n = k\}$ and let the names τ and σ be defined by $\tau = \{\langle \check{n}, p_n \rangle : n \in \omega\}$ and $\sigma = \{\langle \check{n}, p_n^{i_n} \rangle : n \in \omega\}$. Then
(a) $p' \Vdash \tau = \sigma = \check{\emptyset}$;
(b) $p \Vdash \tau = \sigma \triangle \check{S}_1$;
(c) $p \Vdash \sigma = \tau \triangle \check{S}_1$;
(d) $p \Vdash \sigma = \check{\omega} \Leftrightarrow \tau = \check{S}_0$;
(e) $p \Vdash \sigma =^* \check{\omega} \Leftrightarrow \tau =^* \check{S}_0$;
(f) $p \Vdash |\sigma| < \check{\omega} \Leftrightarrow \tau =^* \check{S}_1$.

Theorem 3. Under the assumptions of Lemma 2, White wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ in the game
$\mathcal{G}_1$ iff $\|\sigma \text{ is not equal to } \check{\omega}\| = 1$ iff $p \Vdash \tau \neq \check{S}_0$;
$\mathcal{G}_2$ iff $\|\sigma \text{ is not cofinite}\| = 1$ iff $p \Vdash \tau \neq^* \check{S}_0$;
$\mathcal{G}_3$ iff $\|\sigma \text{ is not supported}\| = 1$ iff $p \Vdash$ "$\tau \cap \check{S}_0$ and $\check{S}_1 \setminus \tau$ are unsupported";
$\mathcal{G}_4$ iff $\|\sigma \text{ is not infinite}\| = 1$ iff $p \Vdash \tau =^* \check{S}_1$.

Proof. We will prove the statement concerning the game $\mathcal{G}_3$ and leave the rest to the reader. So, White wins $\mathcal{G}_3$ iff $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} p_n^{i_n} = 0$, that is, by Lemma 1, $\|\sigma \text{ is not supported}\| = 1$ and the first equivalence is proved.

Let $1 \Vdash$ "σ is not supported" and let G be a $\mathbb{B}$-generic filter over V containing p. Suppose $\tau_G \cap S_0$ or $S_1 \setminus \tau_G$ contains a subset $A \in [\omega]^\omega \cap V$. Then $A \subseteq \sigma_G$, which is impossible.

On the other hand, let $p \Vdash$ "$\tau \cap \check{S}_0$ and $\check{S}_1 \setminus \tau$ are unsupported" and let G be a $\mathbb{B}$-generic filter over V. If $p' \in G$ then, by Lemma 2(a), $\sigma_G = \emptyset$ so σ_G is unsupported. Otherwise $p \in G$ and by the assumption the sets $\tau_G \cap S_0$ and $S_1 \setminus \tau_G$ are unsupported. Suppose $A \subseteq \sigma_G$ for some $A \in [\omega]^\omega \cap V$. Then $A = A_0 \cup A_1$, where $A_0 = A \cap S_0 \cap \tau_G$ and $A_1 = A \cap S_1 \setminus \tau_G$, and at least one of these sets is infinite. But from Lemma 2(c) we have $A_0 = A \cap S_0$ and $A_1 = A \cap S_1$, so $A_0, A_1 \in V$. Thus either $S_0 \cap \tau_G$ or $S_1 \setminus \tau_G$ is a supported subset of ω, which is impossible. So σ_G is unsupported and we are done. $\Box$

In the same way one can prove the following statement concerning Black.

Theorem 4. Under the assumptions of Lemma 2, Black wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ in the game

$\mathcal{G}_1$ iff $\|\sigma$ is equal to $\check{\omega}\| > 0$ iff $\exists q \leq p \;\; q \Vdash \tau = \check{S}_0$;

$\mathcal{G}_2$ iff $\|\sigma$ is cofinite $\| > 0$ iff $\exists q \leq p \;\; q \Vdash \tau =^* \check{S}_0$;

$\mathcal{G}_3$ iff $\|\sigma$ is supported $\| > 0$ iff $\exists q \leq p \;\; q \Vdash$ "$\tau \cap \check{S}_0$ or $\check{S}_1 \setminus \tau$ is supported";

$\mathcal{G}_4$ iff $\|\sigma$ is infinite $\| > 0$ iff $\exists q \leq p \;\; q \Vdash \tau \neq^* \check{S}_1$.

Since for each sequence $\langle b_n \rangle$ in a c.B.a. $\mathbb{B}$

$$\bigwedge_{n\in\omega} b_n \leq \liminf b_n \leq \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} b_n \leq \limsup b_n, \tag{1}$$

we have

Proposition 1. Let $\mathbb{B}$ be a complete Boolean algebra. Then

(a) White has a w.s. in $\mathcal{G}_4 \Rightarrow$ White has a w.s. in $\mathcal{G}_3 \Rightarrow$ White has a w.s. in $\mathcal{G}_2 \Rightarrow$ White has a w.s. in $\mathcal{G}_1$.

(b) Black has a w.s. in $\mathcal{G}_1 \Rightarrow$ Black has a w.s. in $\mathcal{G}_2 \Rightarrow$ Black has a w.s. in $\mathcal{G}_3 \Rightarrow$ Black has a w.s. in $\mathcal{G}_4$.

3. The game $\mathcal{G}_2$

Theorem 5. For each complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) $\mathbb{B}$ is not $(\omega, 2)$-distributive;

(b) White has a winning strategy in the game $\mathcal{G}_1$;

(c) White has a winning strategy in the game $\mathcal{G}_2$.

Proof. (a)⇔(b) is proved in [3] and (c)⇒(b) holds by Proposition 1. In order to prove (a)⇒(c) we suppose $\mathbb{B}$ is not $(\omega, 2)$-distributive. Then $p := \|\exists x \subseteq \check{\omega} \;\; x \notin V\| > 0$ and by The Maximum Principle there is a name $\pi \in V^{\mathbb{B}}$ such that

$$p \Vdash \pi \subseteq \check{\omega} \;\wedge\; \pi \notin V. \tag{2}$$

Clearly $\omega = A_0 \cup A \cup A_p$, where $A_0 = \{n \in \omega : \|\check{n} \in \pi\| \wedge p = 0\}$, $A = \{n \in \omega : \|\check{n} \in \pi\| \wedge p \in (0,p)_{\mathbb{B}}\}$ and $A_p = \{n \in \omega : \|\check{n} \in \pi\| \wedge p = p\}$. We also have $A_0, A, A_p \in V$ and

$$p \Vdash \pi = (\pi \cap \check{A}) \cup \check{A}_p. \tag{3}$$

Let $f : \omega \to A$ be a bijection belonging to V and $\tau = \{\langle \check{n}, \|f(n)\check{\ } \in \pi\| \wedge p\rangle : n \in \omega\}$. We prove

$$p \Vdash f[\tau] = \pi \cap \check{A}. \tag{4}$$

Let G be a $\mathbb{B}$-generic filter over V containing p. If $n \in f[\tau_G]$ then $n = f(m)$ for some $m \in \tau_G$, so $\|f(m)\check{\ } \in \pi\| \wedge p \in G$ which implies $\|f(m)\check{\ } \in \pi\| \in G$ and consequently $n \in \pi_G$. Clearly $n \in A$. Conversely, if $n \in \pi_G \cap A$, since f is a surjection there is $m \in \omega$ such that $n = f(m)$. Thus $f(m) \in \pi_G$ which implies $\|f(m)\check{\ } \in \pi\| \wedge p \in G$ and hence $m \in \tau_G$ and $n \in f[\tau_G]$.

According to (2), (3) and (4) we have $p \Vdash \pi = f[\tau] \cup \check{A}_p \notin V$ so, since $A_p \in V$, we have $p \Vdash f[\tau] \notin V$ which implies $p \Vdash \tau \notin V$. Let $p_n = \|f(n)\check{\ } \in \pi\| \wedge p$, $n \in \omega$. Then, by the construction, $p_n \in (0,p)_{\mathbb{B}}$ for all $n \in \omega$.

We define a strategy Σ for White: at the beginning White plays p and, in the n-th move, plays p_n. Let us prove Σ is a winning strategy for White in the game $\mathcal{G}_2$. Let $\langle i_n : n \in \omega\rangle \in {}^{\omega}2$ be an arbitrary play of Black. According to Theorem 3 we prove $p \Vdash \tau \neq^* \check{S}_0$. But this follows from $p \Vdash \tau \notin V$ and $S_0 \in V$ and we are done. □

4. The game $\mathcal{G}_3$

Firstly we give some characterizations of complete Boolean algebras on which White has a winning strategy in the game $\mathcal{G}_3$. To make the formulas more readable, we will write w_φ for $w(\varphi)$. Also, for $i : \omega \to 2$ we will denote $g^i = \{i \upharpoonright n : n \in \omega\}$, the corresponding branch of the tree ${}^{<\omega}2$.

Theorem 6. For a complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) White has a winning strategy in the game $\mathcal{G}_3$ on $\mathbb{B}$;

(b) There are $p \in \mathbb{B}^+$ and $w : {}^{<\omega}2 \to (0,p)_{\mathbb{B}}$ such that

$$\forall i : \omega \to 2 \;\; \bigvee_{A \in [\omega]^\omega} \bigwedge_{n \in A} w_{i \upharpoonright n}^{i(n)} = 0; \tag{5}$$

(c) There are $p \in \mathbb{B}^+$ and $w : {}^{<\omega}2 \to [0,p]_{\mathbb{B}}$ such that (5) holds.

(d) There are $p \in \mathbb{B}^+$ and $\rho \in V^{\mathbb{B}}$ such that

$$\begin{aligned} p \Vdash \rho \subseteq ({}^{<\omega}2)\check{} \wedge \forall \varphi \in ({}^{<\omega}2)\check{}\ (\varphi^\frown \check{0} \in \rho \,\dot{\vee}\, \varphi^\frown \check{1} \in \rho) \\ \wedge \forall i \in (({}^{\omega}2)^V)\check{}\ (\rho \cap \check{g^i} \text{ is unsupported}). \end{aligned} \tag{6}$$

(e) In some generic extension, $V_{\mathbb{B}}[G]$, there is a subset R of the tree ${}^{<\omega}2$ containing either $\varphi^\frown 0$ or $\varphi^\frown 1$, for each $\varphi \in {}^{<\omega}2$, and having unsupported intersection with each branch of the tree ${}^{<\omega}2$ belonging to V.

Proof. (a)⇒(c). Let Σ be a winning strategy for White. Σ is a function adjoining to each sequence of the form $\langle p, p_0, i_0, \dots, p_{n-1}, i_{n-1}\rangle$, where $p, p_0, \dots, p_{n-1} \in \mathbb{B}^+$ are obtained by Σ and $i_0, i_1, \dots, i_{n-1}$ are arbitrary elements of $\{0,1\}$, an element $p_n = \Sigma(\langle p, p_0, i_0, \dots, p_{n-1}, i_{n-1}\rangle)$ of $(0,p)_{\mathbb{B}}$ such that White playing in accordance with Σ always wins. In general, Σ can be a multi-valued function, offering more "good" moves for White, but according to The Axiom of Choice, without loss of generality we suppose Σ is a single-valued function, which is sufficient for the following definition of p and $w : {}^{<\omega}2 \to [0,p]_{\mathbb{B}}$.

At the beginning Σ gives $\Sigma(\emptyset) = p \in \mathbb{B}^+$ and, in the first move, $\Sigma(\langle p\rangle) \in (0,p)_{\mathbb{B}}$. Let $w_\emptyset = \Sigma(\langle p\rangle)$.

Let $\varphi \in {}^{n+1}2$ and let $w_{\varphi\restriction k}$ be defined for $k \leq n$. Then we define $w_\varphi = \Sigma(\langle p, w_{\varphi\restriction 0}, \varphi(0), \dots, w_{\varphi\restriction n}, \varphi(n)\rangle)$.

In order to prove (5) we pick an $i : \omega \to 2$. Using induction it is easy to show that in the match in which Black plays $i(0), i(1), \dots,$ White, following Σ plays $p, w_{i\restriction 0}, w_{i\restriction 1}, \dots$ Thus, since White wins $\mathcal{G}_3$, we have $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} w_{i\restriction n}^{i(n)} = 0$ and (5) is proved.

(c)⇒(b). Let $p \in \mathbb{B}^+$ and $w : {}^{<\omega}2 \to [0,p]_{\mathbb{B}}$ satisfy (5). Suppose the set $S = \{\varphi \in {}^{<\omega}2 : w_\varphi \in \{0,p\}\}$ is dense in the ordering $\langle {}^{<\omega}2, \supseteq\rangle$. Using recursion we define $\varphi_k \in S$ for $k \in \omega$ as follows. Firstly, we choose $\varphi_0 \in S$ arbitrarily. Let φ_k be defined and let $i_k \in 2$ satisfy $i_k = 0$ iff $w_{\varphi_k} = p$. Then we choose $\varphi_{k+1} \in S$ such that $\varphi_k^\frown i_k \subseteq \varphi_{k+1}$. Clearly the integers $n_k = \mathrm{dom}(\varphi_k)$, $k \in \omega$, form an increasing sequence, so $i = \bigcup_{k\in\omega} \varphi_k : \omega \to 2$. Besides, $i \restriction n_k = \varphi_k$ and $i(n_k) = i_k$. Consequently, for each $k \in \omega$ we have $w_{i\restriction n_k}^{i(n_k)} = w_{\varphi_k}^{i_k} = p$. Now $A_0 = \{n_k : k \in \omega\} \in [\omega]^\omega$ and $\bigwedge_{n\in A_0} w_{i\restriction n}^{i(n)} = p > 0$. A contradiction to (5).

So there is $\psi \in {}^{<\omega}2$ such that $w_\varphi \in (0,p)_{\mathbb{B}}$, for all $\varphi \supseteq \psi$. Let $m = \mathrm{dom}(\psi)$ and let v_φ for $\varphi \in {}^{<\omega}2$ be defined by

$$v_\varphi = \begin{cases} w_\psi & \text{if } |\varphi| < m, \\ w_{\psi^\frown(\varphi\restriction(\mathrm{dom}(\varphi)\setminus m))} & \text{if } |\varphi| \geq m. \end{cases}$$

Clearly $v : {}^{<\omega}2 \to (0,p)_{\mathbb{B}}$ and we prove that v satisfies (5). Let $i : \omega \to 2$ and let $j = \psi^\frown(i \restriction (\omega \setminus m))$. Then for $n \geq m$ we have $v_{i\restriction n}^{i(n)} = w_{\psi^\frown(i\restriction(n\setminus m))}^{i(n)} = w_{j\restriction n}^{j(n)}$. Let $A \in [\omega]^\omega$. Then $A \setminus m \in [\omega]^\omega$ and, since w satisfies (5), for the function j defined above we have $\bigwedge_{n\in A\setminus m} w_{j\restriction n}^{j(n)} = 0$, that is $\bigwedge_{n\in A\setminus m} v_{i\restriction n}^{i(n)} = 0$, which implies $\bigwedge_{n\in A} v_{i\restriction n}^{i(n)} = 0$ and (b) is proved.

(b)⇒(a). Assuming (b) we define a strategy Σ for White. Firstly White plays p and $p_0 = w_\emptyset$. In the n-th step, if $\varphi = \langle i_0, \ldots, i_{n-1}\rangle$ is the sequence of Black's previous moves, White plays $p_n = w_\varphi$. We prove that Σ is a winning strategy for White. Let $i : \omega \to 2$ code an arbitrary play of Black. Since White follows Σ, in the n-th move White plays $p_n = w_{i\restriction n}$, so according to (5) we have $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} p_n^{i_n} = 0$ and White wins the game.

(b)⇒(d). Let $p \in \mathbb{B}^+$ and $w : {}^{<\omega}2 \to (0,p)_{\mathbb{B}}$ be the objects provided by (b). Let us define $v_\emptyset = p$ and, for $\varphi \in {}^{<\omega}2$ and $k \in 2$, let $v_{\varphi^\frown k} = w_\varphi^k$. Then $\rho = \{\langle \check{\varphi}, v_\varphi\rangle : \varphi \in {}^{<\omega}2\}$ is a name for a subset of ${}^{<\omega}2$. If $i : \omega \to 2$, then $\sigma^i = \{\langle (i \restriction n)\check{}, v_{i\restriction n}\rangle : n \in \omega\}$ is a name for a subset of g^i and, clearly,

$$1 \Vdash \sigma^i = \rho \cap \check{g^i}. \tag{7}$$

Let us prove

$$\forall i : \omega \to 2 \;\; 1 \Vdash \rho \cap \check{g^i} \text{ is unsupported}. \tag{8}$$

Let $i : \omega \to 2$. According to the definition of v, for $n \in \omega$ we have $w_{i\restriction n}^{i(n)} = v_{i\restriction(n+1)}$ so, by (5), $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} v_{i\restriction(n+1)} = 0$. By (7) we have $v_{i\restriction(n+1)} = \|(i \restriction (n+1))\check{} \in \rho \cap \check{g^i}\|$ and we have $\|\exists A \in (([\omega]^\omega)^V)\check{}\ \forall n \in A\ (i \restriction (n+1))\check{} \in \rho \cap \check{g^i}\| = 0$ that is $\|\neg\exists B \in (([{}^{<\omega}2]^\omega)^V)\check{}\ B \subset \rho \cap \check{g^i}\| = 1$ and (8) is proved. Now we prove

$$\forall \varphi \in {}^{<\omega}2 \;\; p \Vdash \check{\varphi}^\frown\check{0} \in \rho \;\; \dot{\vee} \;\; \check{\varphi}^\frown\check{1} \in \rho. \tag{9}$$

If $p \in G$, where G is a $\mathbb{B}$-generic filter over V, then clearly $|G \cap \{w_\varphi, p \setminus w_\varphi\}| = 1$. But $w_\varphi = w_\varphi^0 = v_{\varphi^\frown 0} = \|\check{\varphi}^\frown\check{0} \in \rho\|$ and $p \setminus w_\varphi = w_\varphi^1 = v_{\varphi^\frown 1} = \|\check{\varphi}^\frown\check{1} \in \rho\|$ and (9) is proved.

(d)⇒(c). Let $p \in \mathbb{B}^+$ and $\rho \in V^{\mathbb{B}}$ satisfy (6). In V for each $\varphi \in {}^{<\omega}2$ we define $w_\varphi = \|(\varphi^\frown 0)\check{} \in \rho\| \wedge p$ and check condition (c). So for an arbitrary $i : \omega \to 2$ we prove

$$\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} w_{i\restriction n}^{i(n)} = 0. \tag{10}$$

According to (6) for each $n \in \omega$ we have $p \Vdash ((i \restriction n)^\frown 0)\check{} \in \rho \;\dot{\vee}\; ((i \restriction n)^\frown 1)\check{} \in \rho$, that is $p \leq a_0 \vee a_1$ and $p \wedge a_0 \wedge a_1 = 0$, where $a_k = \|((i \restriction n)^\frown k)\check{} \in \rho\|$, $k \in \{0,1\}$, which clearly implies $p \wedge a_0' = p \wedge a_1$, i.e.

$$p \wedge \|((i \restriction n)^\frown 0)\check{} \in \rho\|' = p \wedge \|((i \restriction n)^\frown 1)\check{} \in \rho\|. \tag{11}$$

Let us prove
$$w_{i\restriction n}^{i(n)} = \|(i \restriction (n+1))\check{} \in \rho\| \wedge p. \qquad (12)$$
If $i(n) = 0$, then $w_{i\restriction n}^{i(n)} = \|((i \restriction n)^\frown 0)\check{} \in \rho\| \wedge p = \|((i \restriction n)^\frown i(n))\check{} \in \rho\| \wedge p$ and (12) holds. If $i(n) = 1$, then according to (11) $w_{i\restriction n}^{i(n)} = p \setminus w_{i\restriction n} = p \wedge \|((i \restriction n)^\frown 0)\check{} \in \rho\|' = p \wedge \|((i \restriction n)^\frown 1)\check{} \in \rho\| = p \wedge \|((i \restriction n)^\frown i(n))\check{} \in \rho\|$ and (12) holds again.

Now $\bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} w_{i\restriction n}^{i(n)} = p \wedge \|\exists A \in (([\omega]^\omega)^V)\check{}\ \forall n \in A\ \check{i} \restriction (n+1) \in \rho\| = p \wedge \|\rho \cap \check{g^i}$ is supported$\| = 0$, since by (6) $p \leq \|\rho \cap \check{g^i}$ is unsupported$\|$. Thus (10) is proved.

(d)$\Rightarrow$(e) is obvious and (e)$\Rightarrow$(d) follows from The Maximum Principle. $\square$

Concerning condition (e) of the previous theorem we note that in [5] the following characterization is obtained.

Theorem 7. White has a winning strategy in the game $\mathcal{G}_4$ on a c.B.a. $\mathbb{B}$ if and only if in some generic extension, $V_{\mathbb{B}}[G]$, there is a subset R of the tree ${}^{<\omega}2$ containing either $\varphi^\frown 0$ or $\varphi^\frown 1$, for each $\varphi \in {}^{<\omega}2$, and having finite intersection with each branch of the tree ${}^{<\omega}2$ belonging to V.

Theorem 8. Let $\mathbb{B}$ be a complete Boolean algebra. If forcing by $\mathbb{B}$ produces an independent real in some generic extension, then White has a winning strategy in the game $\mathcal{G}_3$ played on $\mathbb{B}$.

Proof. Let $p = \|\exists x \subseteq \check{\omega}\quad x$ is independent$\| > 0$. Then, by The Maximum Principle there is a name $\tau \in V^{\mathbb{B}}$ such that
$$p \Vdash \tau \subseteq \check{\omega}\ \wedge\ \forall A \in (([\omega]^\omega)^V)\check{}\ \ (|A \cap \tau| = \check{\omega}\ \wedge\ |A \setminus \tau| = \check{\omega}). \qquad (13)$$
Let us prove that $K = \{n \in \omega : \|\check{n} \in \tau\| \wedge p \in \{0,p\}\}$ is a finite set. Clearly $K = K_0 \cup K_p$, where $K_0 = \{n \in \omega : p \Vdash \check{n} \notin \tau\}$ and $K_p = \{n \in \omega : p \Vdash \check{n} \in \tau\}$. Since $p \Vdash \check{K}_0 \subseteq \check{\omega} \setminus \tau\ \wedge\ \check{K}_p \subseteq \tau$, according to (13) the sets K_0 and K_p are finite, thus $|K| < \omega$.

Let $q \in (0,p)_{\mathbb{B}}$ and let p_n, $n \in \omega$, be defined by
$$p_n = \begin{cases} q & \text{if } n \in K, \\ \|\check{n} \in \tau\| \wedge p & \text{if } n \in \omega \setminus K. \end{cases}$$
Then for $\tau_1 = \{\langle \check{n}, p_n\rangle : n \in \omega\}$ we have $p \Vdash \tau_1 =^* \tau$ so according to (13)
$$p \Vdash \tau_1 \subseteq \check{\omega}\ \wedge\ \forall A \in (([\omega]^\omega)^V)\check{}\ \ (|A \cap \tau_1| = \check{\omega}\ \wedge\ |A \setminus \tau_1| = \check{\omega}). \qquad (14)$$

Then $p_n = \|\check{n} \in \tau_1\| \in (0,p)_{\mathbb{B}}$ and we define a strategy Σ for White: at the beginning White plays p and, in the n-th move, White plays p_n.

We prove Σ is a winning strategy for White. Let $\langle p, p_0, i_0, p_1, i_1, \ldots\rangle$ be an arbitrary play in which White follows Σ and let $S_k = \{n \in \omega : i_n = k\}$, for $k \in \{0,1\}$. Suppose $q = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} p_n^{i_n} > 0$. Now $q \leq p$ and $q = \bigvee_{A\in[\omega]^\omega}(\bigwedge_{n\in A\cap S_0} \|\check{n} \in \tau_1\| \wedge \bigwedge_{n\in A\cap S_1}(p \wedge \|\check{n} \notin \tau_1\|) = p \wedge \bigvee_{A\in[\omega]^\omega} \|\check{A} \cap \check{S}_0 \subseteq \tau_1 \wedge \check{A} \cap \check{S}_1 \subseteq \check{\omega} \setminus \tau_1\| \leq \|\exists A \in (([\omega]^\omega)^V)\check{}\ (\check{A} \cap \check{S}_0 \subseteq \tau_1 \wedge \check{A} \cap \check{S}_1 \subseteq \check{\omega} \setminus \tau_1)\|$.

Let G be a $\mathbb{B}$-generic filter over V containing q. Then there is $A \in [\omega]^\omega \cap V$ such that $A \cap S_0 \subseteq (\tau_1)_G$ and $A \cap S_1 \subseteq \omega \setminus (\tau_1)_G$. But one of the sets $A \cap S_0$ and $A \cap S_1$ must be infinite and, since $p \in G$, according to (14), it must be split by $(\tau_1)_G$. A contradiction. Thus $q = 0$ and White wins the game. $\Box$

Theorem 9. Let $\mathbb{B}$ be an $(\omega, 2)$-distributive complete Boolean algebra. Then

(a) If $\langle p, p_0, i_0, p_1, i_1, \ldots\rangle$ is a play satisfying the rules given in Definition 1, then Black wins the game $\mathcal{G}_3$ iff Black wins the game $\mathcal{G}_4$.

(b) Black has a winning strategy in the game $\mathcal{G}_3$ iff Black has a winning strategy in the game $\mathcal{G}_4$.

Proof. (a) The implication "$\Rightarrow$" follows from the proof of Proposition 1(b). For the proof of "$\Leftarrow$" suppose Black wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots\rangle$ in the game $\mathcal{G}_4$. Then, by Theorem 4 there exists $q \in \mathbb{B}^+$ such that $q \Vdash$"σ is infinite". Since the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive we have $1 \Vdash \sigma \in V$, thus $q \Vdash \sigma \in (([\omega]^\omega)^V)\check{}$ and hence $\neg 1 \Vdash$ "σ is not supported" so, by Theorem 4, Black wins $\mathcal{G}_3$.

(b) follows from (a). $\Box$

5. Indeterminacy, problems

Theorem 10. $\diamondsuit$ implies the existence of a Suslin algebra on which the games $\mathcal{G}_1$,$\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are undetermined.

Proof. Let $\mathbb{B}$ be the Suslin algebra mentioned in (c) of Theorem 2. According to Proposition 1(b) and since Black does not have a winning strategy in the game $\mathcal{G}_4$, Black does not have a winning strategy in the games $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ as well. On the other hand, since the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive, White does not have a winning strategy in the game $\mathcal{G}_1$ and, by Proposition 1(a), White does not have a winning strategy in the games $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ played on $\mathbb{B}$. $\Box$

Problem 1. According to Theorem 8, Proposition 1 and Theorem 5 for each complete Boolean algebra $\mathbb{B}$ we have:

$$\mathbb{B} \text{ is } \omega\text{-independent} \Rightarrow \text{White has a winning strategy in } \mathcal{G}_3 \Rightarrow \mathbb{B} \text{ is not } (\omega, 2)\text{-distributive.}$$

Can one of the implications be reversed?

Problem 2. According to Proposition 1(b), for each complete Boolean algebra $\mathbb{B}$ we have:

$$\text{Black has a winning strategy in } \mathcal{G}_1 \Rightarrow \text{Black has a winning strategy in } \mathcal{G}_2 \Rightarrow \text{Black has a winning strategy in } \mathcal{G}_3.$$

Can some of the implications be reversed?

We note that the third implication from Proposition 1(b) can not be replaced by the equivalence, since if $\mathbb{B}$ is the Cohen or the random algebra, then Black has a winning strategy in the game $\mathcal{G}_4$ (Theorem 2(b)) while Black does not have a winning strategy in the game $\mathcal{G}_3$, because White has one (the Cohen and the random forcing produce independent reals and Theorem 8 holds).

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